Indian Statistical Institute
Semestral Examination 2011-2012
B.Math Second Year
Analysis IV

Time : 3 Hours Date : 07.05.2012 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Answer all questions.

Q1. (5+5+10=20 marks) Let K_n be the Frejer's kernel and f be a 2π-periodic continuous function.

(i) Prove that for each $n \geq 1$,

$$K_n(x) = \frac{\sin^2 \frac{(n+1)x}{2}}{\sin^2 \frac{x}{2}}.$$

(ii) For $\delta \in (0, \pi)$, prove that

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} K_n(x)\, dx = 0.$$

(iii) Prove that the sequence of functions $\{\sigma_n\}$ converges to f uniformly, where σ_n is the n-th Cesàro sum of the Fourier series of f.

Q2. (10 marks) Let $f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$ be a continuous 2π-periodic function, and assume that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Conclude that $f \equiv 0$.

Q3. (10 marks) Let f be a continuous 2π- and r-periodic function, where r is a rational number. Does it follows that f is constant.

Q4. (20 marks) Let f be a continuously differentiable 2π-periodic function with $f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{inx}$.

(i) Prove that $\hat{f'}(n) = in\hat{f}(n)$ for all $n \in \mathbb{Z}$.

(ii) Suppose $\int_{-\pi}^{\pi} f(x)\, dx = 0$. Prove that

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx \leq \int_{-\pi}^{\pi} |f'(x)|^2\, dx.$$

Q5. (10 marks) Let f be a 2π-periodic, complex-valued, differentiable function such that $|f(x)| = 1$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that $f(x) = e^{imx}$ for some unique $m \in \mathbb{Z}$, and for all $x \in \mathbb{R}$.

Q6. (10 marks) Prove that there exists a subset of $\mathbb{R}$ which is not measurable.

Q7. (10 marks) Let A and B be two measurable subsets of $\mathbb{R}$. Prove that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

Q8. (10 marks) Find the cardinality of the set of all measurable subsets of $\mathbb{R}$.